Ghost Nodes

June 7, 2019

Take an equation defined by a linear PDE operator

$$\mathcal{L}u = f$$

for $u \in \Omega$, a domain. Discretize $\mathcal{L}$ using some finite difference method via a stencil to L. Define this stencil to be a mapping from the discretization of the full domain to the interior. For example,

$$L = \begin{bmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \end{bmatrix}$$

Notice that this is $N \times (N+2)$, those two endpoints are the boundary points.

Now take Robin boundary conditions. $\alpha u(0) + \beta u'(0) = \gamma$. Discretize u', for example, $u'(0) = \frac{u(1)-u(0)}{\Delta x}$. Notice that when you do this, you get a linear relationship $b_1 \cdot u = \gamma$ as a constraint. Do this on both sides. Now you can write the problem as a constrained linear solve: $Lu = f$ subject to the constraints. But since it's linear, you don't have to. Instead, solve the constraint for $u(0)$ and $u(L)$. You get a linear relationship $u_0 = g_0(u_1, u_2, \ldots, \alpha, \beta, \gamma)$. Do the same for u_L. You can now define an affine operator $Qu = Q_a u + Q_b = \bar{u}$ which goes from the interior of u to the full u by appending the boundary points (and clearly, Q is I except at the ends where it's g as a linear relation + constant). Writing it like this, you have an affine system

$$LQu = Au = f$$

for the solution of the PDE. You can solve this in two ways. You can either concretize the operators. To do this, you'd split

$$L(Q_a u + Q_b) = f$$

and then get the concrete $M = LQ_a$ matrix (which is $N \times N$), and then get $r = f - LQ_b$, and then solve the linear system

$$Mu = r$$

using backslash. Or, you can directly solve $Au = f$ using an iterative solver method. You can easily check that an iterative method like GMRES or CG on an affine A will converge to the correct value by looking at it in residual form.