Review Topics

1. Logic
2. **Proofs**
3. Asymptotics

Proofs

Suppose we have a statement we want to show, e.g.:

$$\forall x \in \mathcal{X} : P(x) \Rightarrow Q(x)$$

- To prove the statement, give a sequence of logical deductions to go from assuming $P(x)$ is True to showing $Q(x)$ is True.

- To disprove the statement, give a counterexample (an example $x \in \mathcal{X}$ such that $P(x)$ is True but $Q(x)$ is False).

Proof Techniques

1. Direct proof
2. Proof by contrapositive
3. Proof by contradiction
4. Proof by induction
5. Proof by cases

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1 - Direct Proof

Goal: To prove $P \Rightarrow Q$

Steps:

- Assume P
- ...
- Therefore Q

2 - Proof by Contrapositive

Goal: To prove $P \Rightarrow Q$

Steps:

- Assume $\neg Q$
- ...
- Therefore $\neg P$

Conclusion: $\neg Q \Rightarrow \neg P$, which is equivalent to $P \Rightarrow Q$

2 - Proof by Contrapositive: Example

Problem: Let $d, n \in \mathbb{N}$ and suppose d divides n.
Show that if n is odd, then d is odd.

2 - Proof by Contrapositive: Example

Problem: Let $d, n \in \mathbb{N}$ and suppose d divides n.
Show that if n is odd, then d is odd.

Proof: We prove the contrapositive: If d is even, then n is even.

Because d divides n, we have $n = dk$ for some $k \in \mathbb{N}$.
Because d is even, $d = 2\ell$ for some $\ell \in \mathbb{N}$.
Then $n = dk = 2k\ell$ is also even. □

3 - Proof by Contradiction

Goal: To prove P

Steps:

- Assume $\neg P$
- $\ldots$
- Show Q
- $\ldots$
- Also show $\neg Q$, a contradiction.

Conclusion: $\neg P \Rightarrow Q \wedge \neg Q =$ False. This is equivalent to True $\Rightarrow P$, hence P.

3 - Proof by Contradiction: Example

Problem: Show that $\sqrt{2} \notin \mathbb{Q}$.

3 - Proof by Contradiction: Example

Problem: Show that $\sqrt{2} \notin \mathbb{Q}$.

Proof: Suppose the contrary that $\sqrt{2} \in \mathbb{Q}$.

This means $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$. We can cancel the common factors, so a and b have no common factors bigger than 1.

From $a = \sqrt{2}b$, we square both sides to get $a^2 = 2b^2$. This shows a^2 is even, which implies a is even. Write $a = 2c$ for some $c \in \mathbb{N}$.

Then $a^2 = 4c^2 = 2b^2$, so $b^2 = 2c^2$ is even. This implies b is even.

We have shown a and b are both even. This contradicts the assumption that a and b have no common factors bigger than 1. Therefore, there cannot be such a and b. We conclude that $\sqrt{2} \notin \mathbb{Q}$. $\square$

4 - Proof by Induction

Goal: To prove $\forall\, n \in \mathbb{N}: P(n)$

Steps:

- Base case: Show $P(1)$
- Induction hypothesis: Assume $P(n-1)$ for some $n \geq 2$
- Induction step: Show $P(n)$

$$P(1) \;\Rightarrow\; P(2) \;\Rightarrow\; P(3) \;\Rightarrow\; P(4) \;\Rightarrow\; \ldots$$

5 - Proof by Cases

Goal: To prove $\forall x \in \mathcal{X} : P(x)$

Steps:

- Partition the domain into subsets (cases): $\mathcal{X} = A_1 \cup \cdots \cup A_n$
- For each case $i = 1, \ldots, n$, prove $\forall x \in A_i : P(x)$

Proof Fallacies – 1

Claim: $-2 = 2$

Proof(?): Assume $-2 = 2$.
Squaring both sides, we have $(-2)^2 = 2^2$, or $4 = 4$, which is True.
We conclude that $-2 = 2$, as desired. $\square$

Proof Fallacies – 1

Claim: $-2 = 2$

Proof(?): Assume $-2 = 2$.
Squaring both sides, we have $(-2)^2 = 2^2$, or $4 = 4$, which is True.
We conclude that $-2 = 2$, as desired. $\square$

Issue: We proved $P \Rightarrow$ True. This is meaningless in determining P.

Proof Fallacies – 2

Claim: $1 = 2$

Proof(?): Assume that $x = y$ for some $x, y \in \mathbb{Z}$. Then,

$$\begin{aligned} x^2 - xy &= x^2 - y^2 && \text{(since } x = y\text{)} \\ x(x - y) &= (x + y)(x - y) \\ x &= x + y && \text{(divide both sides by } x - y\text{)} \\ x &= 2x. \end{aligned}$$

Setting $x = y = 1$ yields the claim. □

Proof Fallacies – 2

Claim: $1 = 2$

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Setting $x = y = 1$ yields the claim. $\square$

Issue: We divided by $x - y = 0$.