Doctoral Confirmation

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Minimal Logic

- Positive fragment of classical logic.
- Replace axioms for negation with

$$\neg\varphi := \varphi \to \bot.$$

- $\bot$ is a propositional symbol with no syntactic significance.
- Deduction through introduction and elimination rules for '$\vee$', '$\wedge$', '$\to$', '$\exists$', '$\forall$'.

$$\text{Minimal} \xrightarrow{+EFQ} \text{Intuitionistic} \xrightarrow{+LEM} \text{Classical}$$

Reverse Mathematics

Key idea: Consider theorems of classical logic which do not hold in minimal logic.

- What is necessary to prove these theorems? What do they imply?
- Build a hierarchy.

Simple example: LEM $\vdash$ WLEM

To show that the other direction is not derivable, we require semantics.

Kripke Semantics

Models logic as a partial order of *states*.

- Each state is labelled with the propositional symbols known to hold at that state.
- A state *may* be labelled with $\bot$. Such a state is *abnormal*.
- Labelling is upwards-closed.
- Define a *forcing* relation $\Vdash$ between states and formulae.
- Will see that forcing is also upwards-closed.
- A formula holds in a given model if it is forced at every state.

Kripke Semantics: Forcing Relation

$A \Vdash P \vee Q$ if $A \Vdash P$ or $A \Vdash Q$.

$A \Vdash P \wedge Q$ if $A \Vdash P$ and $A \Vdash Q$.

$A \Vdash P \to Q$ if for every state $B \succeq A$,

$$B \Vdash P \implies B \Vdash Q.$$

Recall $\neg P \equiv (P \to \bot)$.

$A \Vdash \neg P$ if for every state $B \succeq A$,

$$B \Vdash P \implies B \Vdash \bot.$$

Kripke Semantics: Separation

If there is a model in which a formula φ is not forced, then that formula must not be derivable in minimal logic.

If every instance of a formula ψ holds in such a model, then $\{\psi\} \nvdash \varphi$.

Example: LEM is not derivable from WLEM.

Kripke Semantics: Predicates

Predicates are considered over a domain (e.g. $\mathbb{N}$) of *terms*. States are labelled with the terms that can be examined, and which for which terms each predicate holds.

$A \Vdash \exists_x Px$ if $A \Vdash Pt$ for some term $t \in T(A)$.

$A \Vdash \forall_x Px$ if for every state $B \succeq A$,

$$B \Vdash Pt \qquad \text{for every term } t \in T(B).$$

Drinker Paradox

DP: *In every non-empty pub, there is a person (y) such that if they are drinking, everyone in the pub is drinking.*

$$\exists_y(Py \to \forall_x Px)$$

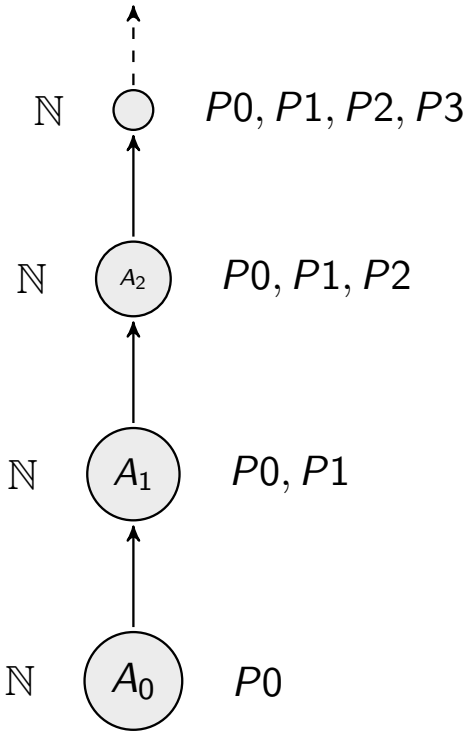
Hε: *In every non-empty pub, there is a person (y) such that if anyone in the pub is drinking, they are drinking.*

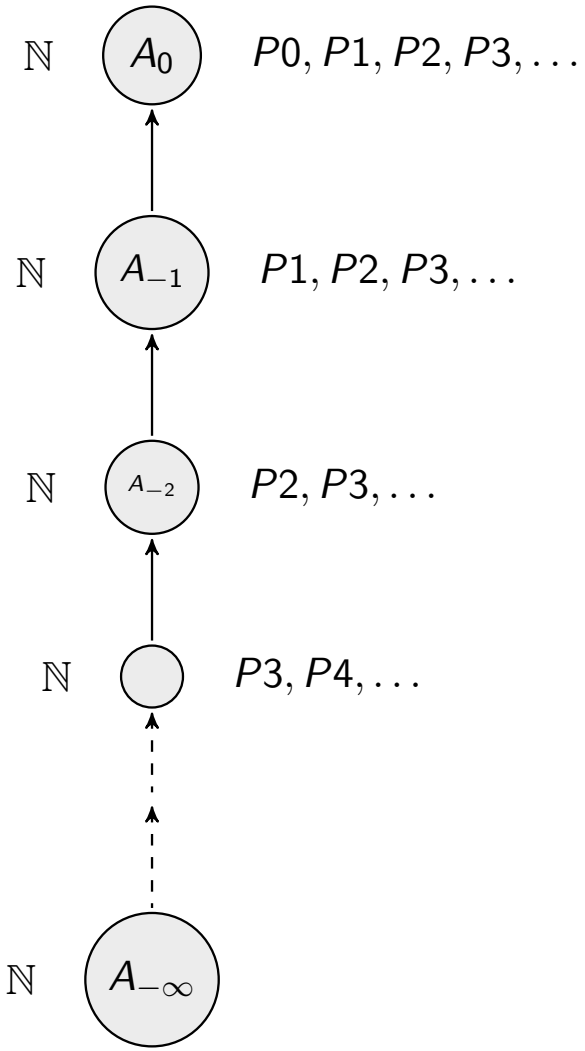
$$\exists_y(\exists_x Px \to Py)$$

Theorem

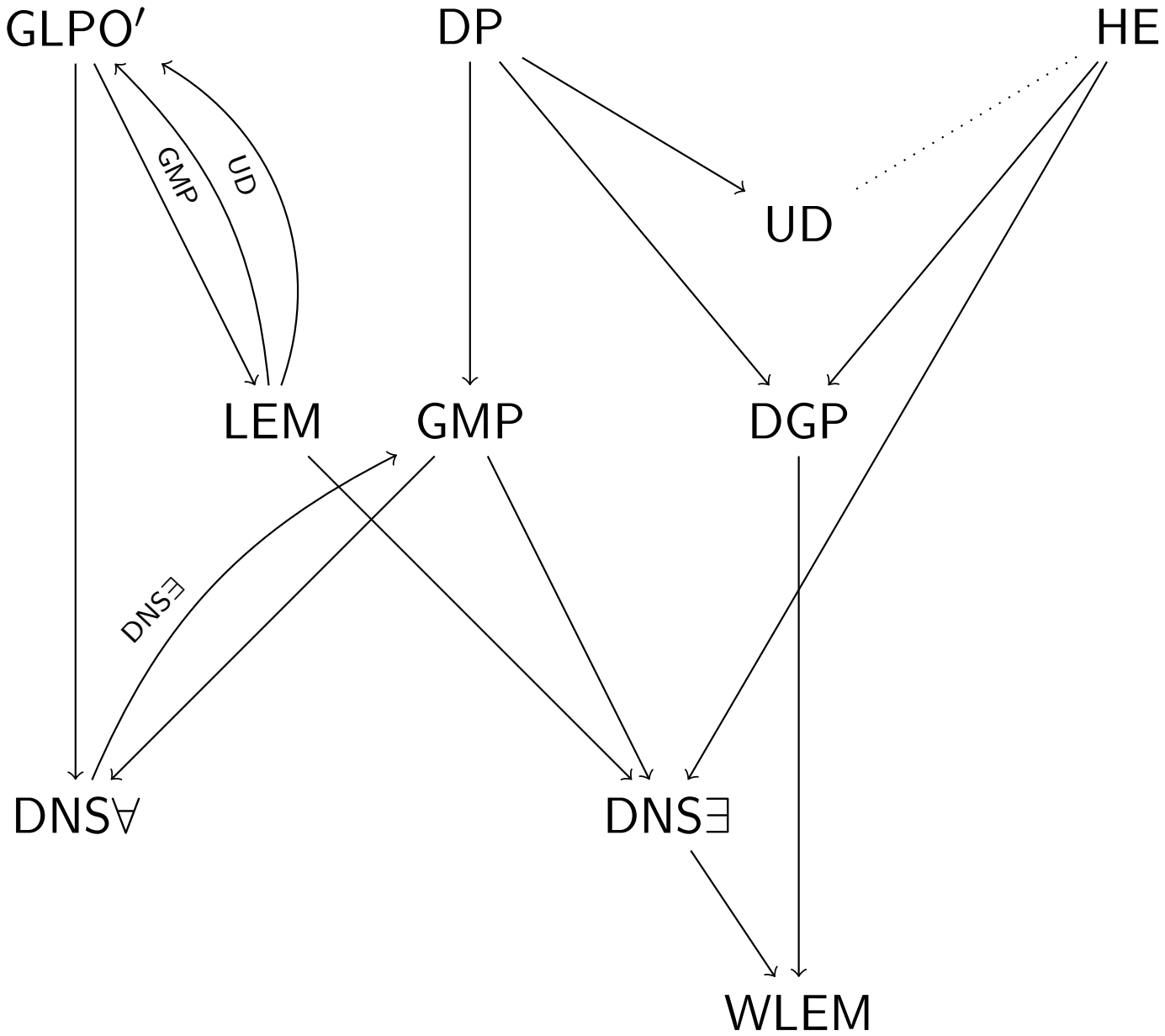
DP $\nvdash$ Hε and Hε $\nvdash$ DP

Drinker Paradox Semantics

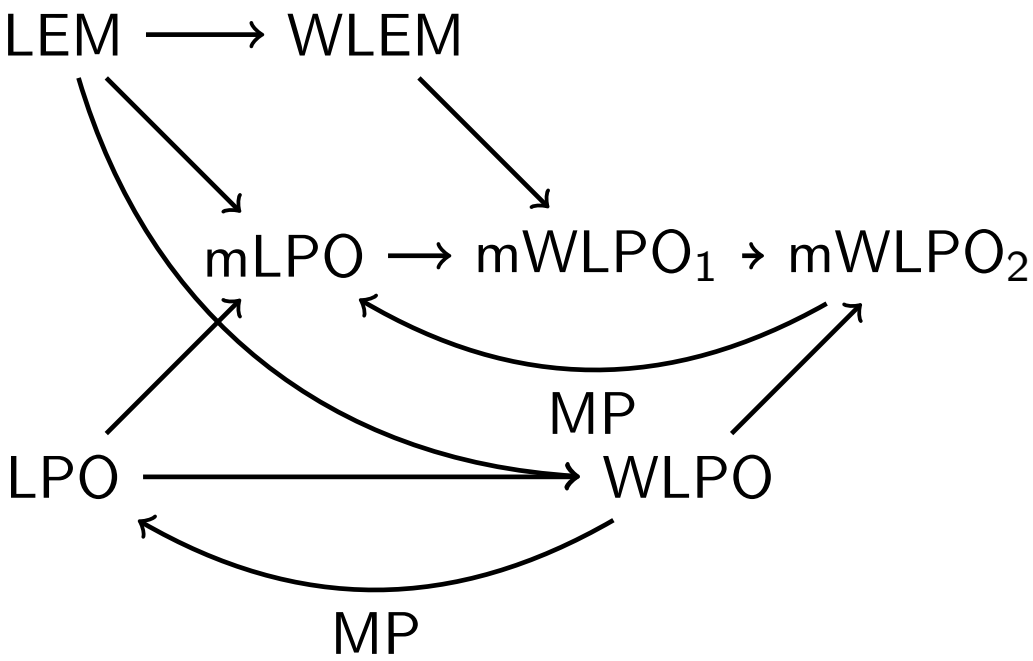




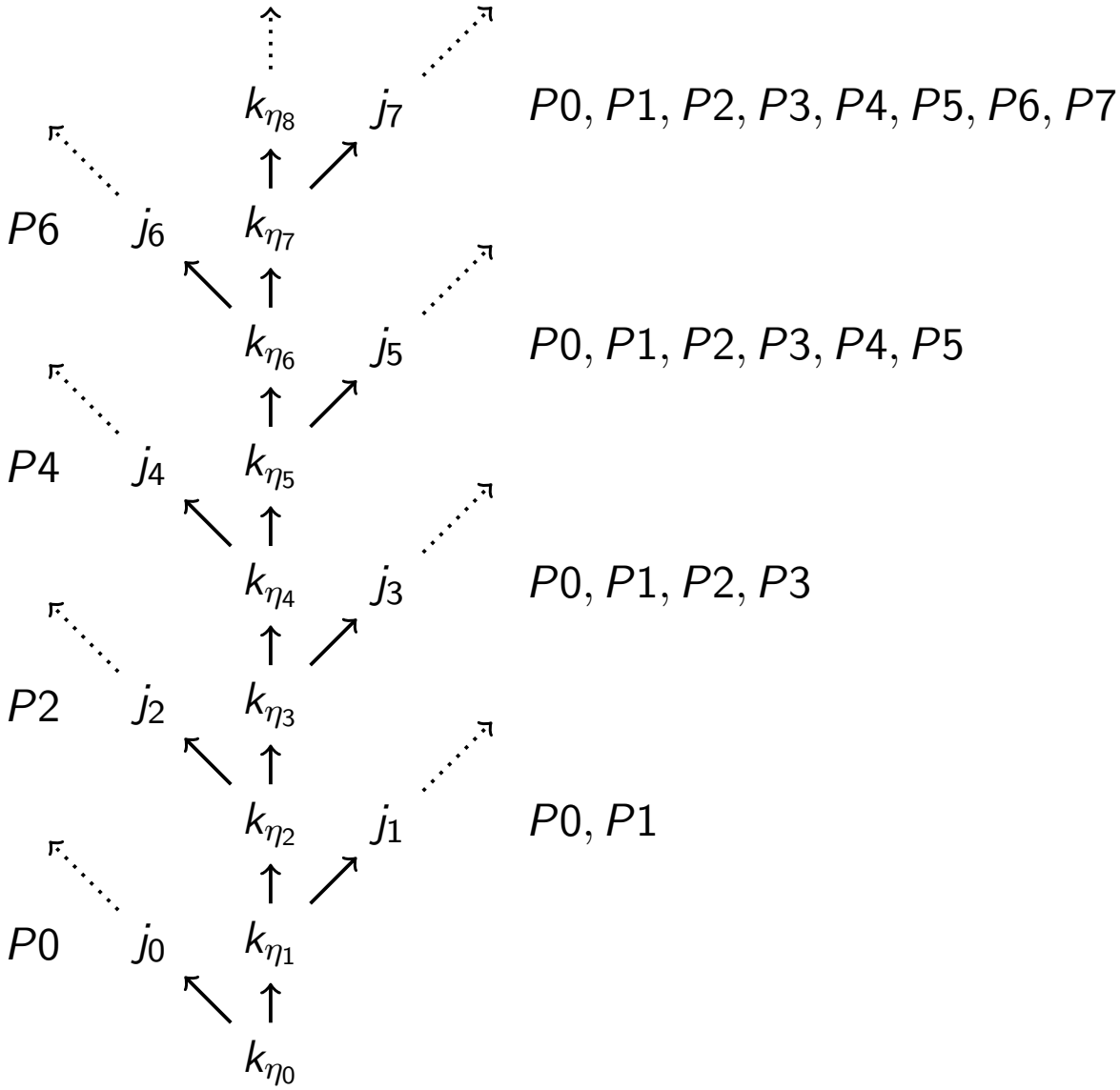
Hierarchy



Sequence Hierarchy



Full Beth Semantics Incompleteness



Minimal Analysis

- Avoid negative definitions.
- Properties of the natural numbers remain decidable.
- Rational numbers require axiom $\forall_n(S(n) = 0 \to S(0) = 0)$.
- Basic approach: replace deductions $\bot \vdash \varphi$ with $(0 = 1) \vdash \varphi$.